# Complex Numbers

David W.H. Swenson

Despite the name, "imaginary" numbers are very important in the physical sciences. Many problems in quantum mechanics require complex numbers in order to solve them, and for most problems complex numbers provide us with better methods.

## 1 The Complex Plane

### 1.1 Real and Imaginary Parts as Cartesian Coordinates

One very important idea about complex numbers is that of the complex plane. You've probably already seen the real numbers represented as a "number line." Each real number is described by a single coordinate.

We can then describe a complex number $z$ with two coordinates: one for the real part ($\Re(z)$), and one for the imaginary part ($\Im(z)$). We usually represent this with the real part along the $x$-axis, and the imaginary part along the $y$-axis. So the complex number $3 + 4i$ is equivalent to the point $(3, 4)$.

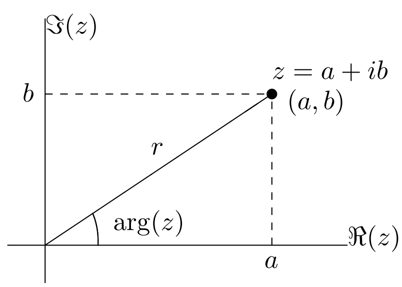


Figure 1: The Complex Plane

**Exercise 1.** What Cartesian point is equivalent to the complex number $6i$? What about $-2$?

### 1.2 Modulus and Argument as Polar Coordinates

In the standard 2D plane, polar coordinates label a point using a distance from the origin $r$ and an angle $\theta$ from the $x$-axis. Just as we used Cartesian coordinates to represent the complex plane in the previous subsection, we will now use polar coordinates. The distance $r$ is called the modulus, or magnitude, and is represented as $|z|$ for the complex number $z$. The angle is called the argument of the number, and is frequently referred to as $\arg(z)$.

**Exercise 2.** For the complex number $z = a + ib$, what is $|z|$ in terms of $a$ and $b$? [Hint: think back to trigonometry.]

**Exercise 3.** For $z = a + ib$, what is $\arg(z)$ in terms of $a$ and $b$? For the special case of a real number ($b = 0$) what is $\arg(z)$?

**Exercise 4.** For a complex number $z$ with modulus $r$ and argument $\theta$, what are $a$ and $b$ such that $z = a + ib$?

Complex numbers can be used to do Euclidean plane geometry. To learn more about this, look into Tristan Needham's book *Visual Complex Analysis.*

# 2 Complex Conjugates

The conjugate of a complex number $z$ is denoted by either $z^*$ or $\bar{z}$. It is the number such that $zz^* = |z|^2$. There is a very simple rule to find the complex conjugate of any complex number: simply put a negative sign in front of any $i$ in the number. Thus $3+4i$ has $3-4i$ as its complex conjugate, and the complex conjugate of $e^{-i\pi/3}$ is $e^{i\pi/3}$.

**Exercise 5.** Show that $(a + ib)(a - ib) = a^2 + b^2$

**Exercise 6.** Show that $(r\ e^{i\theta})(r\ e^{-i\theta}) = r^2$

**Exercise 7.** What is the complex conjugate of a real number?

**Exercise 8.** What is the geometric meaning of the complex conjugate? In other words, start by taking a point in the complex plane. In the Cartesian picture, how does the act of taking the complex conjugate move the point? What about in the polar coordinate picture?

# 3 Euler's Formula

Euler's formula is a very important relation which connects the Cartesian complex plane to the polar complex plane. Euler's formula is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1}$$

This equation can be derived from the power series expansions for the functions $e^x$, $\sin(x)$ and $\cos(x)$.

**Exercise 9** (Advanced)**.** Prove Euler's formula. [Hint : what's the Taylor series (or MacLaurin series, actually) of $e^x$? So what if you replace $x$ by $i\theta$ (remembering that $i^{2n} = (-1)^n$)? Now what are the series expansions for $\cos(\theta)$ and $\sin(\theta)$?]

If we multiply each side of Euler's formula be $r = |z|$, we get $r\,e^{i\theta} = r\,(\cos(\theta) + i\sin(\theta))$. The right size should look familiar from exercise 4. So we can use $r\,e^{i\theta}$ to represent any complex number $z$ with modulus $|z| = r$ and argument $\arg(z) = \theta$. As we showed in exercise 6, when we multiply the complex number $r\ e^{i\theta}$ by its complex conjugate, we get $r^2$, which is independent of $\theta$. Since all the dependence of the argument (*i.e.*angle in the polar plane) of $z = r\ e^{i\theta}$ is contained in the $e^{i\theta}$ term, we refer to this term as a "phase factor."

**Exercise 10.** Using Euler's Formula, show that the simple rule for complex conjugation gives the same results in either real/imaginary form or modulus/argument form. [Hint: take a complex number $z = re^{i\theta}$ and define $a$ and $b$ such that $re^{i\theta} = a + ib$. Then take the complex conjugate.]

**Exercise 11.** Two other formulae are often grouped in with Euler's formula. They are:

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \tag{2}$$

and

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \tag{3}$$

Prove these using Euler's formula as given in equation 1. [Hint: $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.]

**Exercise 12** (Advanced)**.** There's a famous formula in mathematics which combines several of the most important mathematical constants: $e$, $\pi$, $i$, and 1. Construct a formula which is equal to zero, using each of those constants once in your expression. [Hint : remember that $\theta$ in $e^{i\theta}$ is in radians.]

# 4 Powers and Roots of Complex Numbers

Although explicit formulae for powers and roots exist for complex numbers written as the sum of their real and imaginary parts, it is often easier to calculate them using Euler's formula. Namely, to find $z^x$, we first write $z$ as $r\ e^{i\theta}$ and then use

$$z^x = \left(r\ e^{i\theta}\right)^x = r^x e^{i\theta x}$$

When looking for $n$th roots, remember that $\sqrt[n]{z} = z^{1/n}$, and use the same procedure.

**Exercise 13.** What is the square root of $i$?

**Exercise 14.** Prove de Moivre's formula,

$$\left(\cos(\theta) + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)$$

where $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. [Hint : $(e^b)^c = e^{bc}$]

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**Exercise 15** (Advanced)**.** The technique described above can be used to find many trigonometric identities. By first taking the trig function, then using the formulae given by equations 2 and 3, doing some math with the result, and then converting them back to trigonometric forms, you can rather easily obtain many results from trigonometry. As an example, try

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

(To the real show-offs: try $\int \mathrm{d}x \, \sin^2(ax) \, \cos^2(ax) = -\frac{1}{32a} \sin(4ax) + \frac{x}{8}$)