Complex Numbers : Solutions

David W.H. Swenson

Exercise 1. What Cartesian point is equivalent to the complex number $6i$? What about -2?

Since $6i = 0 + 6i$, we identify $a = 0$ and $b = 6$ in $a + bi$. Therefore (a, b) becomes $(0, 6)$. Similarly, -2 gives us $-2 + 0i$, and the point $(-2, 0)$.

Exercise 2. For the complex number $z = a + ib$, what is $|z|$ in terms of a and b? [Hint: think back to trigonometry.]

If we look at figure 1, we see that $|z|$, which is marked r in the figure, is the hypotenuse of a right triangle with side lengths a and b. So by the Pythagorean theorem, we have $|z| = r = \sqrt{a^2 + b^2}$.

Exercise 3. For $z = a + ib$, what is $\arg(z)$ in terms of a and b? For the special case of a real number ($b = 0$) what is $\arg(z)$?

This is again just a little trigonometry. We know that $\arg(z)$ is the angle identified in figure 1, and we again use the fact that we have a right triangle. With b as the length of the side opposite the angle, and a the side adjacent, we have $\tan(\arg(z)) = b/a$, or $\arg(z) = \tan^{-1}(b/a)$.

There is a subtlety to the special case of a real number. The obvious answer is that $\arg(z) = \tan^{-1}(0) = 0$. But what about negative numbers? Since r has to be positive (because it is a distance), using $\arg(z) = 0$ only includes the positive numbers. From looking at figure 1, we can determine that we also need to include the possibility $\arg(z) = \pi$.

The reason is that the function $\tan(\theta)$ is π-periodic. So for any $n \in \mathbb{Z}$, we have

$$\tan(\arg(z)) = 0 \Rightarrow \arg(z) = n\pi$$

This means that $\arg(z) = 0$ is only the solution for $n = 0$. Other valid solutions include $\pm\pi, \pm 2\pi, \ldots$. Positive real numbers are covered by even multiples of π (including 0), and negative numbers are covered by odd multiples.

Exercise 4. For a complex number z with magnitude r and argument θ, what are a and b such that $z = a + ib$?

One more flashback of trig: defining $\theta \equiv \arg(z)$ and $r \equiv |z|$, we have $a = r\cos(\theta)$ and $b = r\sin(\theta)$.

Exercise 5. Show that $(a + ib)(a - ib) = a^2 + b^2$

Just a little algebra:

$$\begin{aligned}(a+ib)(a-ib) &= a^2 - iab + iab - i^2b^2 \\ &= a^2 + 0 - (-1)b^2 \\ &= a^2 + b^2\end{aligned}$$

Exercise 6. Show that $(r\ e^{i\theta})(r\ e^{-i\theta}) = r^2$

Remember that when you multiply exponentials, the exponent adds. So we have

$$\begin{aligned}(r\ e^{i\theta})(r\ e^{-i\theta}) &= r^2 e^{i\theta} e^{-i\theta} \\ &= r^2 e^{i\theta - i\theta} \\ &= r^2 e^0 \\ &= r^2\end{aligned}$$

Exercise 7. What is the complex conjugate of a real number?

For a real number, we can write $z = a + 0i = a$ for some real number a. So the complex conjugate $z^* = a - 0i = a$, which is also equal to z. So a real number is its own complex conjugate. [Suggestion : show this using Euler's $z = r\ e^{i\theta}$ representation of complex numbers.]

Exercise 8. What is the geometric meaning of the complex conjugate? In other words, start by taking a point in the complex plane. In the Cartesian picture, how does the act of taking the complex conjugate move the point? What about in the polar coordinate picture?

In the Cartesian picture, we have $a+ib$, which becomes $a-ib$. This is equivalent to taking the point (a, b) and moving it to $(a, -b)$, which is a reflection across the x-axis. So in the Cartesian picture, complex conjugation is a reflection across the x-axis.

In the polar coordinate picture, we change an angle from positive to negative. So in the polar coordinate picture, complex conjugation changes the angle from counter-clockwise to clockwise.

You should think about these two pictures, and convince yourself that they are equivalent.

Exercise 9 (Advanced)**.** Prove Euler's formula. [Hint : what's the Taylor series (or MacLaurin series, actually) of e^x? So what if you replace x by $i\theta$ (remembering that $i^{2n} = (-1)^n$)? Now what are the series expansions for $\cos(\theta)$ and $\sin(\theta)$?]

We begin with the MacLaurin series of e^x:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

Now we make the change of variables $x \mapsto i\theta$:

$$
\begin{aligned}
e^{i\theta} &= \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{(i\theta)^{2j}}{(2j)!} + \frac{(i\theta)^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \frac{i^{2j}\theta^{2j}}{(2j)!} + \frac{i^{2j+1}\theta^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \frac{-1^j\theta^{2j}}{(2j)!} + i\frac{i^{2j}\theta^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} -1^j \frac{\theta^{2j}}{(2j)!} + i(-1)^j \frac{\theta^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} -1^j \frac{\theta^{2j}}{(2j)!} + i \sum_{j=0}^{\infty} -1^j \frac{\theta^{2j+1}}{(2j+1)!} \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}
$$

The first step splits the sum into even and odd terms. The rest is just manipulation until the last step identifies the series expansions found as those of $\sin(\theta)$ and $\cos(\theta)$.

Strictly speaking, we have to prove that all these series converge over what is called "an infinite radius of convergence." But we'll leave that problem to the folks who have taken complex analysis.

Exercise 10. Using Euler's Formula, show that the simple rule for complex conjugation gives the same results in either real/imaginary form or magnitude/argument form. [Hint: take a complex number $z = re^{i\theta}$ and define a and b such that $re^{i\theta} = a + ib$. Then take the complex conjugate.]

Using the results from exercise 4, for z we have $a = r\cos(\theta)$ and $b = \sin(\theta)$ in Cartesian (real/imaginary) form. For $z^* = r\ e^{-i\theta}$, $a = r\cos(-\theta) = r\cos(\theta)$ and $b = r\sin(-\theta) = -r\sin(\theta)$ in Cartesian form. Comparing these, we see that a for z equals a for z^* and b for z equals $-b$ for z^*. □

Exercise 11. Two other formula are often grouped in with Euler's formula. They are:

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

and

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Prove these using Euler's formula as given in equation (1). [Hint: $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.]

The trick is to use Euler's formula twice. For the positive angle, we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

and for the negative angle, we have

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos(\theta) - i\sin(\theta) \end{aligned}$$

where the second step comes from the parity (even/odd-ness) of the sin and cos functions (given in the hint).

Now all we have to do is either add or subtract the functions. If we add them, we find

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= (\cos(\theta) + i\sin(\theta)) + (\cos(\theta) - i\sin(\theta)) \\ &= 2\cos(\theta) \end{aligned}$$

From that, we get $\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos(\theta)$.

On the other hand, if we subtract them, we find

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= (\cos(\theta) + i\sin(\theta)) - (\cos(\theta) - i\sin(\theta)) \\ &= 2i\sin(\theta) \end{aligned}$$

And from there we easily obtain $\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \sin(\theta)$.

Exercise 12 (Advanced)**.** There's a famous formula in mathematics which combines several of the most important mathematical constants: e, π, i, and 1. Construct a formula which is equal to zero, using each of those constants once in your expression. [Hint : remember that θ in $e^{i\theta}$ is in radians.]

We'll go straight to the answer: $e^{i\pi} + 1 = 0$. A friend bought me a pin with this formula on it. Convince yourself that it is true.

Exercise 13. What is the square root of i?

Following the methodology outlined in the text, we first convert i to Euler's notation. It has modulus 1 and argument $\pi/2$. So

$$\sqrt{i} = \sqrt{e^{i\pi/2}}$$

Now we use the fact that $\sqrt{z} = z^{1/2}$ and we have

$$\begin{aligned} \sqrt{i} &= \left(e^{i\pi/2}\right)^{1/2} = \pm e^{i\pi/2\frac{1}{2}} \\ &= \pm e^{i\pi/4} \end{aligned}$$

If you forget to take both the "positive" and "negative" square roots, you would just end up with $e^{i\pi/4}$. See below to convert that into Cartesian notation.

Continuing with both roots, we exploit the fact that $\pm 1 = e^{i(\pi/2 \mp \pi/2)}$ to obtain

$$\begin{aligned}\sqrt{i} &= e^{i(\pi/2 \mp \pi/2)} e^{i\pi/4} \\ &= e^{i(\pi/2 \mp \pi/2 + \pi/4)} \\ &= e^{i(3\pi/4 \mp \pi/2)} \\ &= e^{i\pi/4} \text{ or } e^{5\pi/4}\end{aligned}$$

Converting these back to real part/imaginary part notation:

$$e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

and

$$e^{5i\pi/4} = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

This exercise is part of an interesting subject in mathematics called the nth roots of unity.

Exercise 14. Prove de Moivre's formula,

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$

where $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. [Hint : $(e^b)^c = e^{bc}$]

Once we have Euler's formula, this is pretty straightforward. Thanks to Euler, we have

$$\begin{aligned}(\cos(\theta) + i \sin(\theta))^n &= \left(e^{i\theta}\right)^n \\ &= e^{(i\theta)n} = e^{i(\theta n)} \\ &= \cos(n\theta) + i \sin(n\theta)\end{aligned}$$

This also reminds us of another important rule when dealing with exponentials, which was given in the hint.

Exercise 15 (Advanced)**.** The technique described above can be used to find many trigonometric identities. By first taking the trig function, then using the formulae given by equations 2 and 3, doing some math with the result, and then converting them back to trigonometric forms, you can rather easily obtain many results from trigonometry. As an example, try

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

(To the real show-offs: try $\int \mathrm{d}x \, \sin^2(ax) \, \cos^2(ax) = -\frac{1}{32a} \sin(4ax) + \frac{x}{8}$)

As the exercise suggests, we first replace the sin and cos functions, then do some arithmetic:

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 \\
&= \frac{1}{-4}\left(e^{i\theta} - e^{-i\theta}\right)^2 + \frac{1}{4}\left(e^{i\theta} + e^{-i\theta}\right)^2 \\
&= \frac{1}{4}\left(-\left(e^{i\theta} - e^{-i\theta}\right)^2 + \left(e^{i\theta} + e^{-i\theta}\right)^2\right) \\
&= \frac{1}{4}\left(-\left(e^{2i\theta} - 2 + e^{-2i\theta}\right) + \left(e^{2i\theta} + 2 + e^{-2i\theta}\right)\right) \\
&= \frac{1}{4}\left(-e^{2i\theta} + 2 - e^{-2i\theta} + e^{2i\theta} + 2 + e^{-2i\theta}\right) \\
&= \frac{1}{4}4 = 1 \quad \square
\end{aligned}$$

Now, for the show-off version. I'll do the math a little less explicitly, but you should be able to at least follow the gist:

$$\begin{aligned}
\int \mathrm{d}x\, \sin^2(ax)\cos^2(ax) &= \int \mathrm{d}x \left(\frac{1}{2i}\left(e^{iax} - e^{-iax}\right)\right)^2 \left(\frac{1}{2}\left(e^{iax} + e^{-iax}\right)\right)^2 \\
&= \int \mathrm{d}x\, -\frac{1}{16}\left(e^{2iax} - 2 + e^{-2iax}\right)\left(e^{2iax} + 2 + e^{-2iax}\right) \\
&= \int \mathrm{d}x\, -\frac{1}{16}\left(e^{4iax} - 2 + e^{-4iax}\right) \\
&= -\frac{1}{16}\left(\int \mathrm{d}x\, e^{4iax} - \int \mathrm{d}x\, 2 + \int \mathrm{d}x\, e^{-4iax}\right) \\
&= -\frac{1}{16}\left(\frac{1}{4ia}e^{4iax} - 2x + \frac{1}{-4ia}e^{-4iax}\right) \\
&= -\frac{1}{16}\frac{1}{4ia}\left(e^{4iax} - e^{-4iax}\right) + \frac{x}{8} \\
&= -\frac{1}{32a}\sin(4ax) + \frac{x}{8} \quad \square
\end{aligned}$$