Differential Equations

David W.H. Swenson

1 Differential Equations

A differential equation is an equation which involves one or more derivatives of an unknown function. From algebra, you're used to solving for unknown variables in known functions. With differential equations, it is the function you're looking for, not the variable.

Differential equations play an absolutely fundamental role in the physical sciences because we are often looking at how some measured quantity changes, either in time or in response to changing an experimental condition (*e.g.*, temperature). Let's consider a quantity that changes in time. The rate of change in time is, of course, just the time derivative of the function. Sometimes we can use our knowledge of that rate to determine a general function. For example, Newton's second law tells us that force is proportional to the second derivative of position with respect to time (which we know as acceleration). This is an example of a differential equation.

There are several categories of differential equations (ordinary/partial, linear/nonlinear, first order, second order, etc.) The boundary value techniques we will discuss in section 2 are relevant to all types of differential equations, although the number of boundary conditions needed will vary. The example differential equations which will be given in section 3 are the ones of most relevance to this class, and are first- and second-order linear ordinary differential equations.

A differential equation can allow more than one solution. Each possible solution is called a "particular solution" of the differential equation. With linear differential equations (which is what we'll be dealing with in this class) a linear combination of all the linearly independent particular solutions is the "general solution" that we want. What this means is that if we can find all the independent particular solutions $Y_j(x)$ to a differential equation, then the general solution will be

$$y(x) = \sum_j A_j Y_j(x)$$

where the A_j are constants to be determined by boundary conditions on the problem.

2 Boundary Value Problems

In physical systems, the constants in the general solution are often determined by "boundary values" or by "initial conditions." Frequently, the function or its derivative must have a certain value either at zero or at infinity, which sets the value of the constants. When dealing with boundary value problems, remember that in order to completely specify the function you need as many boundary conditions as arbitrary constants.

Initial conditions are a frequently used kind of boundary condition which give a specific value to $f(0)$ or $f'(0)$, etc. To find values for the parameters in the general solution, we just solve the system of equations determined by the boundary conditions. (In case the terminology is unfamiliar, "solving a system of equations" is what you did in algebra when you had 2 equations and 2 unknowns, except here we have some arbitrary number of equations, called boundary conditions, and the same number of unknowns, called parameters to the general solution.)

Exercise 1. If the general solution to a differential equation is $y(x) = Ax^2 + Bx + C$ and it is subject to the initial conditions $y(0) = 5$, $y'(0) = 2$, and $y''(0) = 1$, then what are A, B, and C?

Exercise 2. Let's assume that we've already found that the general solution to a differential equation is $y(x) = A\cos(\lambda x) + B\sin(\lambda x)$. Now we will subject this system to several boundary conditions:

1. If the system is constrained by the boundary condition $y(0) = 0$, which variable is fixed, and what does $y(x)$ become?

2. Using your the new form of $y(x)$, which variable is fixed by the boundary condition $y(L) = 0$ for an arbitrary L? [Hint : you *don't* want to end up with the "trivial" solution of $y(x) = 0$.] What is now the new solution for $y(x)$?

3. Now think carefully about your second answer. Did you account for every possible value for which $y(L) = 0$? Since our initial function is an oscillatory function, there will be an infinite number of these. Introduce an integer variable n such that you account for all the possibilities. Write down the correct (possible correct*ed*) form of $y(x)$.

4. You should only have one parameter still undetermined from the original general solution. Let's fix that parameter by requiring that it satisfy the boundary condition $\int_0^L \mathrm{d}x\, y^2(x) = 1$. What is your end result for $y(x)$?

[Hint: take another look at this exercise when you deal with the particle in a box.]

3 Some Specific Differential Equations

3.1 $y' = ay$

One of the simplest differential equations is also one which is frequently encountered in the physical sciences. It simply asks the question, "for what function is the rate of change proportional to the original function?" :

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay \tag{1}$$

By being a little loose with our math,[1] we can make the following set of transformations:

$$\begin{aligned}
\frac{\mathrm{d}y}{y} &= \mathrm{d}x\, a \\
\int \mathrm{d}y\, \frac{1}{y} &= \int \mathrm{d}x\, a \\
\ln(y) &= ax + c_1 \\
y &= c\ e^{ax}
\end{aligned}$$

where $c = e^{c_1}$. So a solution to $y' = ay$ is $y = c\ e^{ax}$.

Exercise 3. Confirm that $y = c\ e^{ax}$ satisfies the differential equation $y' = ay$. That is, plug the solution function into the differential equation and show that it works.

Exercise 4. Does this solution also work if a is negative?

3.2 $y' = f(x)\ y$

This is really just a generalization of the previous section. The general solution is

$$y(x) = c\ e^{\int \mathrm{d}x\, f(x)} \tag{2}$$

Exercise 5. Using the method from the previous section, find this solution. Then verify it by testing the solution in the original differential equation.

3.3 $y'' = -ay$

As you may remember from calculus, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(\alpha x) = -\alpha^2 \sin(\alpha x)$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(\alpha x) = -\alpha^2 \cos(\alpha x)$. (If that is unfamiliar to you, take a moment to review why that is the case.) Comparing this with the differential equation we're trying to solve, $y'' = -ay$, we can identify $a = \alpha^2$. So two solutions to this differential equation are

$$y(x) = \sin(\sqrt{a}\, x)$$

[1] I say "loose" because we haven't shown that we can multiply by differentials like this.

and

$$y(x) = \cos(\sqrt{a}\, x)$$

As discussed previously, the general solution to a differential equation is a linear combination of the specific solutions. So the general solution in this case is:

$$y(x) = A\cos(\sqrt{a}\, x) + B\sin(\sqrt{a}\, x) \tag{3}$$

But what if $a < 0$? In that case, our differential equation is $y'' = by$, where $-a = b > 0$. By analogy with section 3.1, we can see that one solution to this is $y(x) = c\, e^{\sqrt{b}\, x}$.

Exercise 6. Verify that $y(x) = c\, e^{\sqrt{a}\, x}$ is a solution to $y'' = ay$. What about $y'' = -ay$? What is α such that $y(x) = c\, e^{\alpha x}$ is a solution to $y'' = -ay$?

Exercise 7. Show that the solution $y(x) = c\, e^{\sqrt{-a}\, x}$ is not linearly indepedent of the solution $y(x) = A\cos(\sqrt{a}\, x) + B\sin(\sqrt{a}\, x)$. That is, find A and B such that the two solutions are equal.

Exercise 8. Verify that $y(x) = A\cos(\sqrt{a}\, x) + B\sin(\sqrt{a}\, x)$ is a solution to the differential equation $y'' = -ay$. Now what about $y'' = ay$? Find α such that $y(x) = A\cos(\alpha x) + B\cos(\alpha x)$ satisfies $y'' = ay$.

Thanks to the results from the exercises, we can say that the general solution to the differential equation $y'' = -ay$ is given by

$$y(x) = A\cos(\sqrt{a}\, x) + B\sin(\sqrt{a}\, x) \tag{4}$$

Technically, we need to prove that there are no other linearly independent solutions to this differential equation, but we'll leave that problem to the math majors.

Exercise 9. Suppose that we have the differential equation

$$\frac{-\hbar^2}{2m}\Psi'' = E\Psi$$

with the boundary conditions $\Psi(0) = \Psi(L) = 0$. What are allowed values for E? [Hint: this is the particle in a box. See exercise 2.]